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Motion of a Flexible Shallow Spherical Shell in Orbit

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A mathematical model for the motion of a large, flexible shallow spherical shell in a circular orbit is presented. For small elastic displacements and attitude angles the linearized equations for the roll and yaw (out-of-plane) motions completely separate from the pitch (in-plane) and elastic motions. However, the pitch and only the axisymmetric elastic modes are seen to be coupled in the linear range. With the shell's symmetry axis following the local vertical, the structure undergoes a static deformation under the influence of gravity and inertia. Further, the pitch and roll motions are unstable due to the unfavorable moment of inertia distribution. A rigid dumbbell connected to the shell at its apex by a spring-loaded double-gimball joint is proposed to stabilize the structure gravitationally. A sensitivity study of the system response characteristics to the key system parameters is carried out.

Nomenclature

$A_{j,p}$ = arbitrary constant
A_n = modal amplitude
C = longitudinal stiffness factor
$\bar{C}$ = external torques with components (C_x, C_y, C_z)
$C_{j,p}, D_{j,p}$ = constants determined from Eqs. (6.1-6.4) of Ref. 3
$C_y^{(mn)}, C_z^{(mn)}$ = $J_y C_y^{(m)} C_y^{(n)} / M_m \ell^2$ and $J_z C_z^{(m)} C_z^{(n)} / M_m \ell^2$, respectively
$C_y^{(n)}, C_z^{(n)}$ = $\ell \partial \phi_x^{(n)}(0,0)/\partial y$ and $\ell \partial \phi_y^{(n)}(0,0)/\partial z$, respectively
$\bar{c}_c$ = $2\sqrt{\bar{k}_y/(1+c_1)}$ or $2\sqrt{\bar{k}_z/(1+c_2)}$
c_y, c_z = coefficients of viscous damping
$\bar{c}_y, \bar{c}_z$ = $c_y/J_y\omega_c$ and $c_z/J_z\omega_c$, respectively
c_1, c_2 = J_y/I_d and J_z/I_d, respectively
D = flexural rigidity, $Eh^3/12(1-\nu^2)$
$\sum_n \bar{D}^{(n)}$ = torques due to center-of-mass shift effects (zero for unconstrained structures), Eq. (1)
D_n' = term due to center-of-mass shift (zero for unconstrained structures), Eq. (2)
E = Young's modulus
E_n = modal component of external forces
$(\hat{e}_r, \hat{e}, \hat{i})$ = unit vectors in local cylindrical frame
$\bar{G}_R$ = gravitational torques due to rigid body motion, Eq. (1)
$\bar{G}^{(n)}$ = gravitational torques due to elastic motion, Eq. (1)
g_{mn} = gravity coupling between the mth and nth structural modes
g_n = gravity coupling between the rigid body and nth structural mode
h = wall thickness of the shell
I_d = moment of inertia of the dumbbell
I_y = J_y/mR^2
$I_1^{(n)},\ldots,I_7^{(n)}$ = volume integrals in Eqs. (A1-A6)
$\hat{i}, \hat{j}, \hat{k}$ = unit vectors along the principal axes of the undeformed body
J_p, I_p = Bessel function and modified Bessel function of the first kind
J_x, J_y, J_z = principal moments of inertia of the undeformed body
j,p = number of nodal circles and the number of nodal meridians, respectively, in the nth elastic mode
k_y, k_z = torsional spring constants
$\bar{k}_y, \bar{k}_z$ = $k_y/J_y\omega_c^2$ and $k_z/J_z\omega_c^2$, respectively
ℓ = base radius of the shell
Mode (j,p) = elastic mode with j nodal circles and p nodal meridians
$M^{(c)}$ = gravity gradient matrix operator given in Ref. 4
M_{ij} = elements of M matrix given in Ref. 4
M_n = modal mass of nth mode
m = mass of the shell
$\sum_n Q^{(n)}$ = inertia torques due to elastic motion, Eq. (1)
$\bar{q}$ = elastic displacement vector (q_x, q_y, q_z)
R = radius of curvature of the shell
$\bar{R}$ = inertia torque due to rigid body motion of the shell
$\bar{r}$ = instantaneous position vector of generic point in the body
$\bar{r}_0$ = position vector of generic point in the undeformed body (ξ_x, ξ_y, ξ_z)
$\bar{r}_1$ = vector from the mass center of the shell to the origin of (x_c, y_c, z_c) coordinate frame (Fig. 1)
$\bar{r}_2$ = position vector of a generic point on undeformed shell in (x_c, y_c, z_c) coordinate frame (Fig. 1)
r_c = radial distance of the shell element from the symmetry axis of the shell
t = time
β = polar angle (Fig. 1)
β_0 = phase angle
γ, δ = dumbbell deflection angles
δ_m = mass of the concentrated mass placed at the end of diameter of circular plate
δ_{mn} = Kronecker delta
ϵ_n = A_n/ℓ
ζ = nondimensional radial distance $= r_c/\ell$
ζ_d = damping ratio, $\bar{c}_y/\bar{c}_c$ or $\bar{c}_z/\bar{c}_c$
θ, ψ, ϕ = pitch, yaw, and roll angles
ν = Poisson's ratio
$\lambda_{j,p}$ = frequency parameter
ρ = mass density
σ_y, σ_z = small rotations at the origin about y and z axes due to elastic deformations
τ = $\omega_c t$

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$\Phi^{(n)}$ = mode shape vector
φ_{mn} = inertia coupling between the mth and nth structural modes, Eq. (2)
φ_n = inertia coupling between the rigid body modes and nth structural mode, Eq. (2)
$\Omega_x, \Omega_y, \Omega_z$ = $(J_z - J_y)/J_x$, $(J_x - J_z)/J_y$, and $(J_y - J_x)/J_z$, respectively
$\bar{\omega}$ = body angular velocity vector
ω_c = orbit angular velocity
ω_n = natural frequency of nth mode
ω_∞ = $(C/hR^2)^{1/2}$
$(\dot{\ }), (\ )'$ = d( )/dt and d( )/dτ, respectively
$c(\ ), s(\ )$ = cos( ) and sin( ), respectively

Introduction

FUTURE communication satellites may involve the use of very large, shallow dish-type structures to be used as receivers and reflectors of communication signals. Due to its inherent size, the entire structure may have to be considered as flexible in the analysis of the dynamics. A finite-element analysis of such large structures in orbit may require a very large number of elements to be considered in the model even though only a few elastic modes might have a predominant effect on the motion. In this paper an attempt is made to use a continuum approach based on the model developed in Ref. 1 to analyze the motion of an orbiting shallow spherical shell with its axis nominally along the local vertical. (To the authors' knowledge this represents the first such analysis of the dynamics of a shell-type structure in orbit.)

Equations of Motion

Figure 1 shows a shallow spherical shell in orbit with the various symbols defined. The following assumptions were made in deriving the equations of motion:

1) The mass and elastic properties are distributed continuously and uniformly throughout the domain of the shell.
2) The thickness of the shell is small as compared to the height of the shell.
3) The ratio of the height H to the base radius ℓ is much less than unity (condition for shallowness).
4) The shell is completely free (without any constraints).
5) The elastic deformations perpendicular to the symmetry axis (i.e., x axis) of the shell are negligible compared to the deformations parallel to the symmetry axis, i.e., only transverse vibrations are considered.
6) The center of mass of the shell is moving in a circular orbit.
7) The symmetry axis of the shell is nominally along the local vertical.
8) The influence of attitude motion and elastic motion on the orbit is negligible.

The equations of motion of an arbitrary flexible body in orbit have been derived in Refs. 1 and 2 in the following form for small elastic deformations:

Equations of the rigid body rotational modes

$$\bar{R} + \sum_n \bar{Q}^{(n)} + \sum_n \bar{D}^{(n)} = \bar{G}_R + \sum_n \bar{G}^{(n)} + \bar{C} \tag{1}$$

Equations of elastic motion

$$\ddot{A}_n + \omega_n^2 A_n + \frac{\varphi_n}{M_n} + \frac{1}{M_n}\sum \varphi_{mn} = \frac{1}{M_n}\left[g_n + \sum_m g_{mn} + D_n' + E_n\right] \quad (n=1,2,\ldots) \tag{2}$$

Vector expressions for the various terms in Eqs. (1) and (2) can be found in Ref. 1. In using the above two sets of equations to describe the motion of a flexible body, it is assumed that the natural mode shapes and frequencies of the body are known a priori.

The natural mode shapes of the transverse vibrations of a shallow spherical shell can be conveniently expressed in a cylindrical system of coordinates (r_c, β, x_c) defined with respect to the (x_c, y_c, z_c) frame (Fig. 1).[3] Each mode shape is characterized by a nodal pattern consisting of concentric nodal circles centered about the axis of the shell and a set of nodal meridians. Mathematical expressions for the natural frequencies and the mode shapes of the transverse vibrations of a shallow spherical shell with completely free edge have been obtained in Ref. 3. The elastic mode shapes are given by[3]

$$\phi_x^{(n)} = A_{j,p}\left[\frac{\ell^{p+4}}{RD\lambda_{j,p}^4} C_{j,p}\zeta^p + J_p(\lambda_{j,p}\zeta) + D_{j,p}I_p(\lambda_{j,p}\zeta)\right]\cos p(\beta+\beta_0) \tag{3}$$

After expressing the instantaneous position vector $\bar{r}_0$ of a generic point on the shell as

$$\bar{r}_0 = \bar{r}_1 + \bar{r}_2 \tag{4}$$

where

$\bar{r}_1$ = the vector from the center of mass of the shell to the origin of $(x_c y_c z_c)$
$\bar{r}_2$ = the vector from the origin of $(x_c y_c z_c)$ to a generic point on the shell

one can show that[4]

$$\sum_n \bar{Q}^{(n)} = \int_v [2\bar{r}_2 \times (\bar{\omega}\times\dot{\bar{q}}) + \bar{r}_2\times(\dot{\bar{\omega}}\times\bar{q}) + \bar{q}\times(\dot{\bar{\omega}}\times\bar{r}_2) - (\bar{r}_2\cdot\bar{\omega})(\bar{\omega}\times\bar{q}) - (\bar{q}\cdot\bar{\omega})(\bar{\omega}\times\bar{r}_2)]\mathrm{d}m \tag{5}$$

$$\sum_n \bar{G}^{(n)} = \int_v (\bar{r}_2 \times M^{(c)}\bar{q} + \bar{q}\times M^{(c)}\bar{r}_2)\mathrm{d}m \tag{6}$$

$$\varphi_n = \int_v [\Phi^{(n)}\cdot\dot{\bar{\omega}}\times\bar{r}_2 + \Phi^{(n)}\cdot\bar{\omega}\times(\bar{\omega}\times\bar{r}_2)]\mathrm{d}m \tag{7}$$

$$\sum_m \varphi_{mn} = \int_v \Phi^{(n)}\cdot\bar{\omega}\times(\bar{\omega}\times\bar{q})\mathrm{d}m \tag{8}$$

$$g_n = \int_v \Phi^{(n)}\cdot M^{(c)}\bar{r}_2\mathrm{d}m \tag{9}$$

$$\sum_m g_{mn} = \int_v \Phi^{(n)}\cdot M^{(c)}\bar{q}\mathrm{d}m \tag{10}$$

where

$M^{(c)}$ = matrix operator given by Eq. (4.19) of Ref. 4

$$\Phi^{(n)} = \phi_x^{(n)}\hat{i}$$

$$\bar{q} = \sum_n A_n(t)\Phi_x^{(n)}(r_c,\beta)$$

The various vectors in Eqs. (5-10) can be expressed in their component forms as

$$\bar{r}_2 = r_c\hat{e}_r + x_c\hat{i} \tag{11}$$

$$\bar{\omega} = \omega_r\hat{e}_r + \omega_\beta\hat{e}_\beta + \omega_x\hat{i} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k} \tag{12}$$

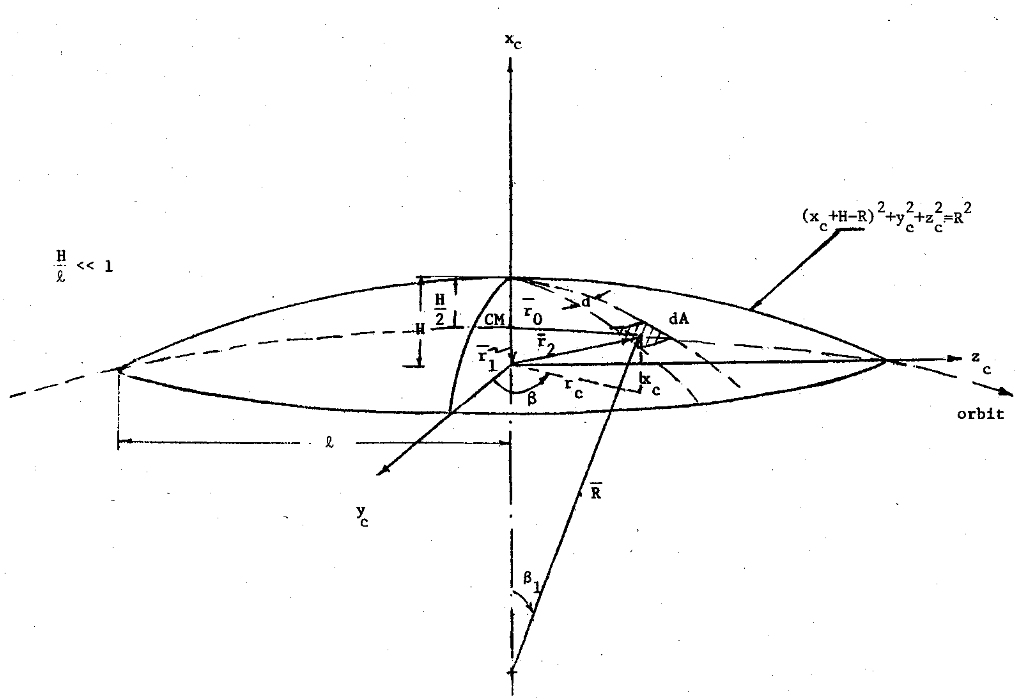

Fig. 1 Shallow spherical shell.

where

$$\hat{e}_r = c\beta\hat{j} + s\beta\hat{k}; \quad \hat{e}_\beta = -s\beta\hat{j} + c\beta\hat{k}$$

$$\omega_r = \omega_y c\beta + \omega_z s\beta; \quad \omega_\beta = -\omega_y s\beta + \omega_z c\beta$$

After substituting the component forms of Eqs. (11) and (12) into Eqs. (5-10), followed by a complex sequence of algebraic manipulations, the expressions for the coupling terms in their component form can be obtained as presented in the Appendix.

The nonlinear equations of motion of the orbiting shallow spherical shell with its axis nominally along the local vertical can be obtained by substituting Eqs. (A1-A6) into Eqs. (1) and (2).[4] In order to examine the stability of the system for small-amplitude initial conditions one can linearize the nonlinear equations assuming small-amplitude pitch θ, roll ϕ, yaw ψ, and elastic displacements A_n. As a result of this one can arrive at the following linear equations of motion for the shallow shell in orbit:

$$\psi'' - \Omega_x\psi - (1+\Omega_x)\phi' = C_x/J_x\omega_c^2 \tag{13}$$

$$\phi'' + 4\Omega_z\phi + (1-\Omega_z)\psi' = C_z/J_z\omega_c^2 \tag{14}$$

$$\theta'' - 3\Omega_y\theta - 2\sum_n \epsilon_n' I_1^{(n)}\ell/J_y = C_y/J_y\omega_c^2 \tag{15}$$

$$\epsilon_n'' + (\Omega_n^2 - 3)\epsilon_n + 2\theta' I_1^{(n)}/M_n\ell = (3I_1^{(n)}/M_n\ell) + E_n/M_n\omega_c^2\ell$$
$$(n=1,2,\ldots) \tag{16}$$

By an examination of Eqs. (13-16) one can conclude (assuming no external torques to be present) that,

1) In the linear range of operation, the motion in the roll-yaw degree of freedom can be studied independently of the pitch and elastic motions.

2) The pitch and elastic motions are coupled directly to each other through their rates.

3) Since $I_1^{(n)} = 0$ for all elastic modes except for the axisymmetric ones (i.e., modes with no nodal meridians) only axisymmetric modes are responsible for the coupling of pitch and the elastic motions. (Roll and yaw motions are also coupled to the elastic motion through the axisymmetric modes by the nonlinear terms only.) Nonaxisymmetric modes are independent of the rigid body rotational motions, ψ, ϕ, and θ.

4) The axisymmetric elastic modes are subjected to a constant excitation force due to the orbital motion and gravity effects.

5) The pitch and the roll-yaw motions are unstable about the present nominal orientation because of the inertia distribution.

In order to stabilize the pitch motion, a passive stabilization procedure using a rigid dumbbell, similar to that proposed in Ref. 5 in the case of flat plates, is considered in the next section.

Gravitational Stabilization

A gravitationally stabilized shallow spherical shell in a circular orbit is shown in Fig. 2. The stabilizing dumbbell is assumed to be rigid and hinged to the shell at its apex by a spring-loaded double-gimbal joint. Thus, the dumbbell has two degrees of freedom with respect to the shell. Damping is assumed at the hinges of the gimbal.

The reaction torques on the shell due to the relative motion between the dumbbell and the shell are given by

$$C_x = [k_z(\delta - \sigma_z) + c_z(\dot{\delta} - \dot{\sigma}_z)]s\gamma$$
$$C_y = k_y(\gamma - \sigma_y) + c_y(\dot{\gamma} - \dot{\sigma}_y)$$
$$C_z = [k_z(\delta - \sigma_z) + c_z(\dot{\delta} - \dot{\sigma}_z)]c\gamma \tag{17}$$

and the modal components of the reaction torques in Eq. (16) are expressed as

$$E_n = C_y \frac{\partial \phi_x^{(n)}}{\partial z}\bigg|_{\substack{y=0\\z=0}} + C_z \frac{\partial \phi_x^{(n)}}{\partial y}\bigg|_{\substack{y=0\\z=0}} \tag{18}$$

By assuming small elastic rotations of the shell at the hinge connection, σ_y and σ_z can be expressed as

$$\sigma_y = \sum_n \epsilon_n C_z^{(n)}; \quad \sigma_z = \sum_n \epsilon_n C_y^{(n)} \tag{19}$$

After substituting Eqs. (17-19) into Eqs. (13-16), one arrives at the following linear equations of motion of the shell for small amplitude pitch θ, roll ϕ, yaw ψ, γ, δ, and elastic motions:

$$\psi'' - \Omega_x \psi - (1+\Omega_x)\phi' = 0 \tag{20}$$

$$\phi'' + 4\Omega_z \phi + (1-\Omega_z)\psi' = \bar{c}_z \delta' + \bar{k}_z \delta - \sum_n (\bar{c}_z \epsilon_n' + \bar{k}_z \epsilon_n) C_y^{(n)} \tag{21}$$

$$\theta'' - 3\Omega_y \theta - 2\sum_n \epsilon_n' I_1^{(n)} \frac{\ell}{J_y} = \bar{c}_y \gamma' + \bar{k}_y \gamma - \sum_n (\bar{c}_y \epsilon_n' + \bar{k}_y \epsilon_n) C_z^{(n)} \tag{22}$$

$$\epsilon_n'' + (\Omega_n^2 - 3)\epsilon_n + 2\theta' \frac{I_1^{(n)}}{M_n \ell} = \frac{3I_1^{(n)}}{M_n \ell} + (\bar{c}_y \gamma' + \bar{k}_y \gamma) \frac{J_y}{M_n \ell^2} C_z^{(n)}$$
$$+ (\bar{c}_z \delta' + \bar{k}_z \delta) \frac{J_z}{M_n \ell^2} C_y^{(n)} - \sum_m (\bar{c}_y \epsilon_m' + \bar{k}_y \epsilon_m) C_z^{(mn)}$$
$$- \sum_m (\bar{c}_z \epsilon_m' + \bar{k}_z \epsilon_m) C_y^{(mn)} \qquad (n = 1,2,\ldots) \tag{23}$$

The linearized equations of motion of the dumbbell are given by

$$\gamma'' + \bar{c}_y (1+c_1)\gamma' + [3 + \bar{k}_y (1+c_1)]\gamma + 3(1+\Omega_y)\theta$$
$$- (1+c_1) \sum_n (\bar{c}_y \epsilon_n' + \bar{k}_y \epsilon_n) C_z^{(n)} + 2\sum_n \epsilon_n' I_1^{(n)} \frac{\ell}{I_d} = 0 \tag{24}$$

$$\delta'' + \bar{c}_z (1+c_2)\delta' + [4 + \bar{k}_z (1+c_2)]\delta + 4(1-\Omega_z)\phi$$
$$- (1-\Omega_z)\psi' - (1+c_2) \sum_n (\bar{c}_z \epsilon_n' + \bar{k}_z \epsilon_n) C_y^{(n)} = 0 \tag{25}$$

Since $C_y^{(n)} = C_z^{(n)} = 0$ for all modes except for the modes with only one nodal meridian, it is evident from Eqs. (20-25) that in the linear range the pitch, roll, yaw, γ, and δ motions are coupled to the elastic motion only through the axisymmetric modes for which $I_1^{(n)} = 0$ and the modes with only one nodal meridian for which $C_y^{(n)} \neq 0$, and $C_z^{(n)} \neq 0$.

Also, it should be noted that the values of the natural frequencies and the mode shape functions of only the axisymmetric modes are modified by the presence of the dumbbell. The modified modal shape functions could be calculated either by a finite-element analysis or by using a series approximation of assumed shape functions. However, it is noted, in the absence of the dumbbell, that the pitch rotational mode is weakly coupled to the axisymmetric elastic modes of the shell. With the addition of the dumbbell end masses, the effect of this coupling is further diminished. An order-of-magnitude analysis for the system parameters involved here shows that, to a good first-order approximation, the axisymmetric modes can be considered independently of all the rigid rotational modes. Furthermore, the characteristics of the lowest frequency (rigid rotational) modes which are studied subsequently, are also found to be virtually independent of the axisymmetric modes. In view of this, the axisymmetric mode shapes as given in Ref. 3 are used here only as a first approximation to the actual mode shapes in order to study the characteristics of the lowest frequency mode. The natural frequencies and mode shapes of the other elastic modes characterized by nodal meridians remain unaffected by the presence of the dumbbell.

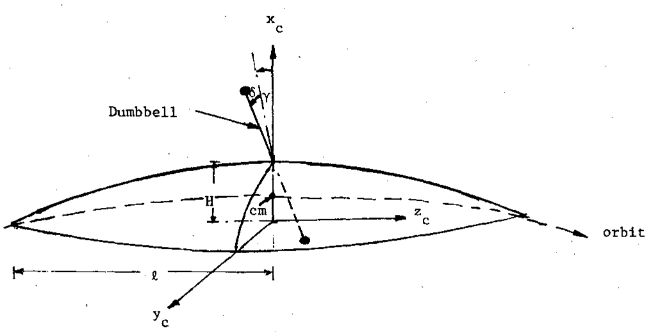

Fig. 2 Shallow spherical shell with dumbbell.

Stability

The stability of the motion of the gravitationally stabilized shallow spherical shell in orbit for small initial conditions can be analyzed by an examination of the roots of the characteristic equation for the system described by Eqs. (20-25). Since an exact analysis would require an infinite number of elastic modes to be considered in the model, the characteristic determinant of the system of Eqs. (20-25) would be of infinite order. However, it is known that for practical purposes one can obtain sufficiently accurate results by retaining only a finite number of elastic modes in the model. The system model thus obtained is called the "truncated model." In this section we consider the stability of a few such truncated models.

Rigid Shallow Spherical Shell

In this case all of the elastic modes of the shell are neglected. This results in the decoupling of pitch and γ motions from roll, yaw, and δ motions. Hence, the stability of the system is governed by the roots of the following two independent characteristic equations:

Roll-yaw-δ

$$s^6 + \alpha_1 s^5 + \alpha_2 s^4 + \alpha_3 s^3 + \alpha_4 s^2 + \alpha_5 s + \alpha_6 = 0 \tag{26}$$

Pitch-γ

$$s^4 + \alpha_7 s^3 + \alpha_8 s^2 + \alpha_9 s + \alpha_{10} = 0 \tag{27}$$

where

$$\alpha_1 = \bar{c}_z (1+c_2)$$
$$\alpha_2 = 5 + \bar{k}_z (1+c_2) - \Omega_z (\Omega_x - 3)$$
$$\alpha_3 = \bar{c}_z [c_2 (1 + 3\Omega_z - \Omega_z \Omega_x) + 4 - \Omega_x]$$
$$\alpha_4 = \bar{k}_z [c_2 (1 + 3\Omega_z - \Omega_z \Omega_x) + 4 - \Omega_x] + 4[1 + 3\Omega_z - 2\Omega_z \Omega_x]$$
$$\alpha_5 = -4\bar{c}_z (c_2 \Omega_z + 1)\Omega_x$$
$$\alpha_6 = -[\bar{k}_z (c_2 \Omega_z + 1) + 4\Omega_z] 4\Omega_x$$
$$\alpha_7 = \bar{c}_y (1+c_1)$$
$$\alpha_8 = \bar{k}_y (1+c_1) + 3(1-\Omega_y)$$
$$\alpha_9 = 3\bar{c}_y (1 - c_1 \Omega_y)$$
$$\alpha_{10} = 3\bar{k}_y (1 - c_1 \Omega_y) - 9\Omega_y;$$
$$-1 \leq \Omega_z < 0, \quad 0 < \Omega_y \leq 1$$

After applying the Routh-Hurwitz criterion for stability, we arrive at the following necessary and sufficient conditions for the stability:

$$\bar{c}_z > 0; \quad \bar{k}_z > [\Omega_z(\Omega_x - 3) - 5]/(1 + c_2)$$

$$c_2 < (4 - \Omega_x)/(\Omega_z\Omega_x - 3\Omega_z - 1)$$

$$\bar{k}_z > -4[1 + 3\Omega_z - 2\Omega_z\Omega_x]/[c_2(1 + 3\Omega_z - \Omega_z\Omega_x) + 4 - \Omega_x]$$

$$\Omega_x < 0; \quad \bar{k}_z > -4\Omega_z/(1 + c_2\Omega_z); \quad \bar{c}_y > 0$$

$$\bar{k}_y > -3(1 - \Omega_y)/(1 + c_1); \quad c_1 < 1/\Omega_y$$

$$\bar{k}_y > 3\Omega_y/(1 - c_1\Omega_y)$$

$$\Delta_i > 0 \qquad (i = 1, \ldots, 5)$$

where the Δ_i's are the principal minors of the determinant

$$\begin{vmatrix} \alpha_1 & 1 & 0 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & 1 & 0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \\ 0 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 \\ 0 & 0 & 0 & \alpha_6 & \alpha_5 \end{vmatrix}$$

It can be seen from these conditions that in addition to the requirement that positive damping be provided by the gimbal assembly, there is a lower boundary on the torsional spring stiffnesses ($\bar{k}_y, \bar{k}_z$). If the shell is assumed to be perfectly axisymmetric and hence $\Omega_x = 0$, then Eq. (26) will have a double root at $s = 0$. This results in an unstable roll-yaw-δ motion. However, this instability can be prevented by introducing a slight asymmetry in the mass distribution of the shell such that $\Omega_x < 0$ as dictated by the stability conditions. This asymmetry may be achieved by placing small concentrated masses at the ends of the meridian along the roll axis of the shell.

Figure 3 shows the loci of the roots of Eq. (26) corresponding to the lowest frequency mode. The following values for the parameters have been assumed in generating Fig. 3 and subsequent figures: damping ratio $\zeta_d = 0.1$, inertia ratios $c_1 = c_2 = 0.9$, $H/\ell = 0.02$, and $H/h = 10$. It is apparent from this figure that by placing heavier concentrated masses at the ends of the diameter along the body roll axis, the roll-yaw stability of the system can be improved. Also, it can be noted that with the increase in the spring stiffness $\bar{k}_z$ the characteristic roots move toward the imaginary axis.

Flexible Shell with a Finite Number of Elastic Modes

If it is assumed that small concentrated masses are placed at the ends of the diameter along the roll axis to achieve roll-yaw stability as mentioned in the above section, then the constant β_0 in Eq. (3) can assume only the following values: $\beta_0 = 0$ and $\pi/2p$, where p = number of nodal meridians ($p \neq 0$).[6] With these concentrated masses present, the natural frequencies and mode shapes of those elastic modes whose nodal lines do not pass through the masses are slightly different (perturbed) as compared with the values obtained for a completely symmetrical homogeneous shell. By assuming, $\delta m/M_n \ll 1$, where δm = mass of the concentrated masses and $M_n = m$ = mass of the shell, one can estimate to a first-order approximation the values of the perturbed natural frequency and mode shapes by the following expressions[6]:

$$\omega_{n_p}^2 \approx \omega_n^2 \left\{1 - \frac{\delta m}{M_n}[\phi_x^{(n)}(\ell)]^2\right\}; \quad \frac{\delta m}{M_n} \ll 1 \tag{28}$$

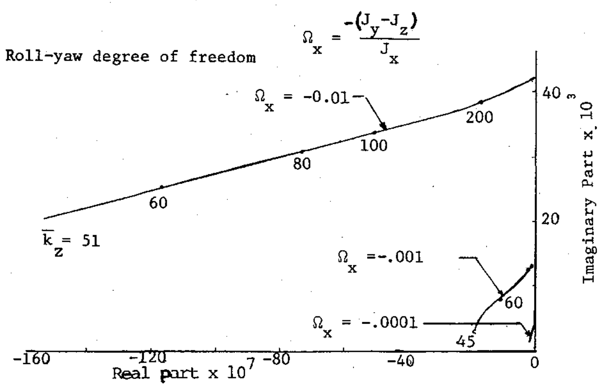

Fig. 3 Root locus of lowest frequency mode of shallow shell with dumbbell for the cases of: a) rigid shell and b) flexible shell with nodal meridian along the pitch axis.

$$\phi_{x_p}^{(n)} \approx \phi_x^{(n)} + \sum_{\substack{m \\ (m \neq n)}} \mu_m \phi_x^{(m)} \tag{29}$$

where

$\omega_{n_p}, \phi_{x_p}^{(n)}$ = perturbed natural frequency and mode shape, respectively
$\omega_n, \phi_x^{(n)}$ = unperturbed natural frequency and mode shape, respectively
δm = total mass added

$$\mu_m = \frac{\omega_n^2}{\omega_m^2 - \omega_n^2}\frac{\delta m}{M_n}\phi_m(\ell)\phi_n(\ell)$$

Earlier it was concluded that in the absence of external forces and torques, only the axisymmetric elastic modes ($p = 0$) and the elastic modes with one nodal meridian ($p = 1$) influence the pitch, roll, yaw, γ, and δ motions. Hence, in the present analysis we do not consider elastic modes with more than one nodal meridian ($p > 1$). When $p = 1$, the constant β_0 in Eq. (3) can assume the values 0 or $\pi/2$.

If we consider a truncated model which has only one elastic mode and that mode is also axisymmetric ($p = 0$), we arrive at the following necessary conditions for the stability of the pitch, γ, and elastic motions within the linear range:

$$\bar{c}_y > 0; \quad \bar{k}_y > \frac{3\Omega_y - \Omega_1^2}{(1 + c_1)}$$

$$\Omega_1^2 - 3 > \left[3c_1(1 + \Omega_y) - \frac{4I_1^{(1)2}}{M_1 I_y \bar{c}_y}\right] \Big/ (1 + c_1)$$

$$\bar{k}_y > \left[9 - 3\Omega_1^2(1 - \Omega_y) - \frac{4I_1^{(1)2} c_1 \bar{c}_y}{M_1 I_y}\right] \Big/ [3(1 - c_1\Omega_y) + (1 + c_1)(\Omega_1^2 - 3)]$$

$$\Omega_1^2 - 3 > -\frac{4I_1^{(1)2}}{M_1 I_y}(c_1\bar{k}_y + 3)/3(1 - c_1\Omega_y)\bar{c}_y$$

$$\bar{k}_y > 3\Omega_y/(1 - c_1\Omega_y)$$

It can be seen from the above conditions that in addition to the requirement that the positive damping be provided, there are lower bounds on the torsional spring stiffness $\bar{k}_y$ and the natural frequency $\omega_1(\Omega_1)$. The stability conditions for the case of the rigid body which were mentioned earlier can be derived from the above conditions after some algebraic manipulations and then taking the limit as $(\Omega_1^2 - 3) \to \infty$.

The stability conditions for the roll, yaw, and δ motions are exactly the same as those for the rigid shallow shell case.

If, on the other hand, we consider a truncated model with only one elastic mode and that mode has one nodal meridian along the pitch axis ($\beta_0=\pi/2$), the necessary conditions for stability of the pitch, γ, and elastic motions result as,

$$\bar{c}_y>0$$

$$\bar{k}_y>-(\Omega_I^2-3\Omega_y)/(1+c_I+C_z^{(1,1)})$$

$$\Omega_I^2-3>[3(1-c_I\Omega_y)-3C_z^{(1,1)}(1-\Omega_y)](1+c_I)$$

$$\bar{k}_y>[9-3\Omega_I^2(1-\Omega_y)]/(\alpha_3/\bar{c}_y)$$

$$\Omega_I^2-3>3\Omega_y C_z^{(1,1)}/(1-\Omega_y c_I)$$

$$\bar{k}_y>9\Omega_y(\Omega_I^2-3)/(\alpha_5/\bar{c}_y)$$

where

$$\alpha_3/\bar{c}_y=(\Omega_I^2-3)(1+c_I)-3(1-c_I\Omega_y)+3C_z^{(1,1)}(1-\Omega_y)$$

$$\alpha_5/\bar{c}_y=3(\Omega_I^2-3)(1-\Omega_y c_I)-9\Omega_y C_z^{(1,1)}$$

The stability conditions for the roll-yaw and δ motion are exactly the same as those of the rigid shallow shell case.

When more than one elastic mode is retained in the model, a formal expansion of the characteristic determinant is algebraically complicated. Hence, we employ a digital algorithm to evaluate the characteristic equation and characteristic roots when all the system parameters are known.

Since for $p=1$, $\beta_0=0$ or $\pi/2$, one encounters the following two situations, depending on the orientation of the nodal meridian and with the presence of other axisymmetric elastic modes:

Situation 1: If $\beta_0=\pi/2$, the nodal meridian will be along the pitch axis of the shell. Hence, from Eq. (3)

$$C_y^{(n)}=0 \text{ for all } n$$

and

$$C_z^{(n)}=\begin{cases}A_{j,I}\lambda_{j,I}(1+C_{j,I})/2, & p=1\\ 0, & p\neq 1\end{cases} \tag{30}$$

This results in the roll ϕ, yaw ψ, and δ motions being completely decoupled from pitch, γ, and elastic motions, i.e., (ψ,ϕ,δ) and $(\theta,\gamma,\epsilon_I,\ldots,\epsilon_n)$ are two independent sets. Thus, the shell behaves like a rigid body in the ψ, ϕ, and δ degrees of freedom.

Figure 4 shows the root loci of the lowest frequency mode of the system. It can be noted from the figure that the influence of the axisymmetric modes on the other system modes is very weak. Since the coupling between the axisymmetric modes and the rigid body modes is very weak, a negligible amount of damping is imparted into the axisymmetric elastic modes. Hence, the characteristic roots corresponding to the axisymmetric modes lie very close to the imaginary axis. We also note from Fig. 4 that with an increase in spring stiffness $\bar{k}_z$ the characteristic roots corresponding to the lowest frequency modes move toward the imaginary axis.

Situation 2: If $\beta_0=0$, the nodal meridian will be along the roll axis of the shell. Hence from Eq. (3)

$$C_z^{(n)}=0$$

and

$$C_y^{(n)}=\begin{cases}A_{j,I}\lambda_{j,I}(1+C_{j,I})/2, & p=1\\ 0, & p\neq 1\end{cases} \tag{31}$$

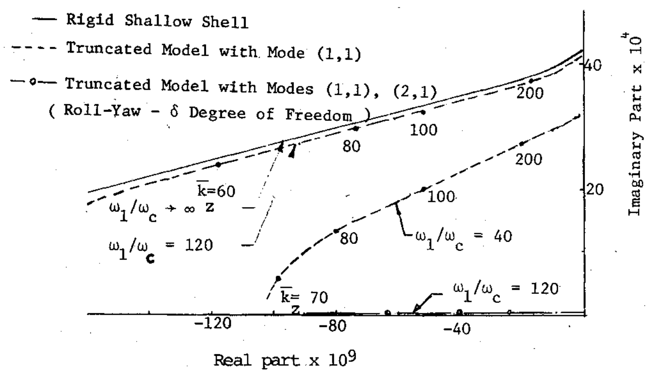

Fig. 4 Root locus of lowest frequency mode of the shallow shell with dumbbell for the case of nodal meridian along the pitch axis [the root loci of lowest frequency mode of pitch-γ degrees of freedom corresponding to the case of nodal meridian along the roll axis lie very close to the curve labeled $(\omega_I/\omega_c)\to\infty$].

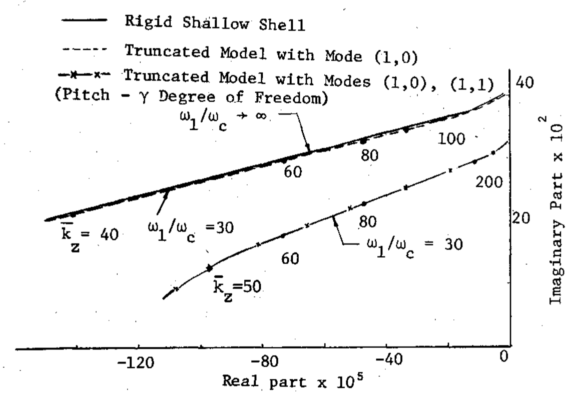

Fig. 5 Root locus of the lowest frequency mode of the shallow shell with dumbbell for the case of nodal meridian along the roll axis.

A study of Eqs. (20-25) reveals that for the present case, the roll ϕ, yaw ψ, δ motion, and elastic modes with only one nodal meridian form an independent set from that of the pitch θ, γ motion, and the axisymmetric elastic modes. Thus, the system stability is governed by two independent characteristic equations. The qualitative conclusions that were made for Fig. 4 with respect to the effect of increase in the values of the torsional spring stiffness and the increase in the number of elastic modes in the model also hold for Fig. 5.

Conclusions

The equations of motion of a shallow spherical shell in orbit with its symmetry axis nominally following the local vertical have been presented in this paper. The structure is gravitationally unstable because of its unfavorable moment of inertia distribution. It is suggested that gravitational stabilization could be achieved by attaching a rigid dumbbell with heavy tip masses in order to provide the favorable composite moment of inertia distribution.

Under the influence of gravity and centrifugal forces, the shallow shell is found to undergo a small static deformation. In the linear range of operation, the pitch motion is weakly coupled to the axisymmetric elastic modes of the shell.

The roll-yaw motion of the shell may be stabilized by adding two concentrated masses at the ends of the meridian along the roll axis.

With the increase in the torsional spring stiffness of the hinge, the system motion tends toward pure oscillations. However, it should also be noted that there is a lower bound on the value of the torsional spring stiffness below which the system stability criteria would not be satisfied. Also, with the increase in the number of elastic modes in the truncated model, the stability of the truncated model deteriorates.

To damp the motion of the system in all its modes, especially the low-frequency modes, active dampers (control systems) are needed. Moreover, it is thought that with the use of the passive gimballed dumbbell stabilization device, together with active controllers, both the peak forces and fuel consumption could be significantly reduced. Such a study would represent a logical extension to the present work.

Appendix

The component forms of the various coupling terms in Eqs. (5-10) are as follows.

$$\bar{Q}^{(n)} = [-2\dot{A}_n(\omega_y I_2^{(n)} + \omega_z I_3^{(n)}) - A_n(\dot{\omega}_y I_2^{(n)} + \dot{\omega}_z I_3^{(n)} + \omega_x\omega_z I_2^{(n)} - \omega_x\omega_y I_3^{(n)})]\hat{i} + [2(\dot{A}_n\omega_y + A_n\dot{\omega}_y)I_1^{(n)} + A_n(-\dot{\omega}_x I_2^{(n)} - 2\omega_z\omega_x I_1^{(n)} + \omega_z\omega_y I_2^{(n)} + (\omega_z^2 - \omega_x^2)I_3^{(n)})]\hat{j} + [2(\dot{A}_n\omega_z + A_n\dot{\omega}_z)I_1^{(n)} - A_n(\dot{\omega}_x I_3^{(n)} - 2\omega_x\omega_y I_1^{(n)} + \omega_y^2 I_2^{(n)} + \omega_y\omega_z I_3^{(n)} - \omega_x^2 I_2^{(n)})]\hat{k} \tag{A1}$$

$$\bar{G}^{(n)} = A_n\{[M_{21}I_3^{(n)} + M_{31}I_2^{(n)}]\hat{i} + [-2M_{21}I_5^{(n)} - 2M_{31}I_4^{(n)} + M_{11}I_3^{(n)} - M_{23}I_6^{(n)} - \frac{M_{22}}{2}(I_7^{(n)} + I_3^{(n)}) + \frac{M_{33}}{2}(I_7^{(n)} - I_3^{(n)})]\hat{j} + [2M_{21}I_4^{(n)} - 2M_{31}I_5^{(n)} - M_{23}I_7^{(n)} + \frac{M_{22}}{2}(I_6^{(n)} + I_2^{(n)}) - \frac{M_{33}}{2}(I_6^{(n)} - I_2^{(n)})]\hat{k}\} \tag{A2}$$

$$\varphi_n = \dot{\omega}_y I_3^{(n)} - \dot{\omega}_z I_2^{(n)} - (\omega_y^2 + \omega_z^2)I_1^{(n)} + \omega_x\omega_y I_2^{(n)} + \omega_x\omega_z I_3^{(n)} \tag{A3}$$

$$\varphi_{mn} = -A_m(\omega_y^2 + \omega_z^2)\delta_{mn}M_n \tag{A4}$$

$$g_n = M_{11}I_1^{(n)} + M_{21}I_2^{(n)} - M_{31}I_3^{(n)} \tag{A5}$$

$$g_{mn} = A_m M_{11}\delta_{mn}M_n \tag{A6}$$

where

$$I_1^{(n)} = \int_v x_c\phi_x^{(n)}\,dm; \qquad I_2^{(n)} = \int_v r_c c\beta\phi_x^{(n)}\,dm$$

$$I_3^{(n)} = \int_v r_c s\beta\phi_x^{(n)}\,dm; \qquad I_4^{(n)} = \int_v x_c c2\beta\phi_x^{(n)}\,dm$$

$$I_5^{(n)} = \int_v x_c s2\beta\phi_x^{(n)}\,dm; \qquad I_6^{(n)} = \int_v r_c c3\beta\,dm$$

$$I_7^{(n)} = \int_v r_c s3\beta\,dm$$

Since the shell is assumed to be completely free, it can be shown that $I_2^{(n)} = I_3^{(n)} = 0$.[4]

By substituting the mode shape functions given in Eq. (3) into the integrands of the integrals $I_1^{(n)}$ through $I_7^{(n)}$ one can arrive at the following results[4]:

$$I_1^{(n)} = 2m\delta_{op}I_{x_0}^{(j)}; \qquad I_4^{(n)} = m\cos p\beta_0\delta_{2p}I_{x_2}^{(j)}$$

$$I_5^{(n)} = -m\sin p\beta_0\delta_{2p}I_{x_2}^{(j)}; \qquad I_6^{(n)} = m\ell\cos p\beta_0\delta_{3p}I_r^{(j)}$$

$$I_7^{(n)} = -m\ell\sin p\beta_0\delta_{3p}I_r^{(j)} \tag{A7}$$

where

m = mass of the shell

$$\delta_{kp} = \begin{cases} 1, & p=k \\ 0, & p\neq k \end{cases} \qquad (k=0,2,3)$$

$$I_{x_0}^{(j)} = 2A_{j,0}\frac{\ell^2}{R}(1+\nu)J_1(\lambda_{j,0})/\lambda_{j,0}^3$$

$$I_{x_2}^{(j)} = \frac{\ell^2}{2R}A_{j,2}\left\{\frac{\ell^6}{12RD\lambda_{j,2}^4}C_{j,2} + \frac{D_{j,2}}{\lambda_{j,2}^3}[2\lambda_{j,2}\{I_0(\lambda_{j,2})+1\} - 8I_1(\lambda_{j,2})] + \frac{1}{\lambda_{j,2}^3}[2\lambda_{j,2}\{J_0(\lambda_{j,2})+1\} - 8J_1(\lambda_{j,2})]\right\}$$

$$I_r^{(j)} = \frac{\ell^7}{6RD\lambda_{j,3}^4}C_{j,3} + \frac{1}{\lambda_{j,3}^3}[8 - 8J_0(\lambda_{j,3}) - 4\lambda_{j,3}J_1(\lambda_{j,3}) - \lambda_{j,3}^2 J_2(\lambda_{j,3})] + \frac{D_{j,3}}{\lambda_{j,3}^3}[\lambda_{j,3}^2 I_2(\lambda_{j,3}) - 4\lambda_{j,3}I_1(\lambda_{j,3}) + 8I_0(\lambda_{j,3}) - 8]A_{j,3}$$

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